Special points and Poincaré bi-extensions

by Daniel Bertrand, with an *Appendix* by Bas Edixhoven

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The context is the following :

i) in a joint project with D. Masser, A. Pillay and U. Zannier [7], we aim at extending to semi-abelian schemes the Masser-Zannier approach [11] to Conjecture 6.2 of R. Pink's preprint [13]; this conjecture also goes under the name "*Relative Manin-Mumford*". Inspired by Anand Pillay's suggestion that the semi-constant extensions of [6] may bring trouble, I found a counter-example, which is described in Section 1 below.

ii) at a meeting in Pisa end of March, Bas Edixhoven found a more concrete way of presenting the counter-example, with the additional advantage that the order of the involved torsion points can be controlled in a precise way : this is the topic of the Appendix.

iii) finally, I realized that when rephrased in the context of mixed Shimura varieties, the construction, far from providing a counter-example, actually *supports* Pink's general Conjecture 1.3 of [13]; a sketch of this view-point is given in Section 2.

1 A counter-example to relative Manin-Mumford ...

This counterexample is provided by a "Ribet section" on a semi-abelian scheme B/X of relative dimension 2 over a base curve X. Roughly speaking, given an elliptic curve $E_0/\mathbb{C}$ *with complex multiplications* and using an idea of L. Breen, K. Ribet constructed a non-torsion point β_0 with strange divisibility properties on any given non isotrivial extension B_0 of E_0 by $\mathbb{G}_m$, cf. [10]. In the relative situation B/X, the very same construction yields:

Theorem 1. *Let B/X be a non constant (hence non isotrivial) extension of $E_0 \times X$ by $\mathbb{G}_m$. There exists a section $\beta : X \to B$ which does not factor through any proper closed subgroup scheme of B/X, but whose image $Y := \beta(X)$ meets the torsion points of the various fibers of B/X infinitely often (so, Zariski-densely, since X is a curve).*

More precisely, let X be a smooth connected affine curve defined over (say) $\mathbb{C}$, with function field $K := \mathbb{C}(X)$. We may have to delete some points of X, or consider finite covers of X, but will still denote by X the resulting curve. We write x for the generic point

of X, i.e. $K = \mathbb{C}(x)$, and $\xi \in X(\mathbb{C})$ for its closed points. We start with a CM elliptic curve $E_0/\mathbb{C}$, denote by $\hat{E}_0 \simeq E_0$ its dual, and fix an antisymmetric isogeny

$$\varphi : \hat{E}_0 \to E_0.$$

This means that its transpose $\hat{\varphi} : \hat{E}_0 \to E_0$ is equal to $-\varphi$, i.e., in the identification $\hat{E}_0 \simeq E_0$, that φ is a totally imaginary complex multiplication.

We consider the constant elliptic schemes $E = E_0 \times_{\mathbb{C}} X, \hat{E} = \hat{E}_0 \times_{\mathbb{C}} X$, and fix a non constant section $q : X \to \hat{E}$. In particular, q is not a torsion section, and the semi-abelian scheme B/X attached to q is a non constant[1], hence non isotrivial, extension of E/X by $\mathbb{G}_m$; conversely, any such B is of this type. Let $\pi : B \to E$ be the corresponding X-morphism. Extending the construction of [10], we will attach to these data (q, φ) a section β of B/X such that the section $p := \pi \circ \beta$ of E/X satisfies $p = 2\,\varphi \circ q$. Furthermore, β will have the following "*lifting property*" : *for any $\xi \in X(\mathbb{C})$ such that $p(\xi)$ is a torsion point on the fiber $E_\xi \simeq E_0$ of $E \to X$, its lift $\beta(\xi)$ is automatically a torsion point on the fiber B_ξ of $B \to X$.*

Before describing this construction, let us show that such a section β does satisfy the conditions announced in Theorem 1 :

• on the one hand, since $q \in \hat{E}(X)$ has infinite order, the only proper closed subgroup schemes of B projecting onto X are contained in finite unions of translates of $\mathbb{G}_m \times_{\mathbb{C}} X$ and of fibers of the projection; and the section $p = 2\,\varphi \circ q \in E(X)$ too has infinite order. Therefore, β, which projects to p on E, cannot factor through any proper closed subgroup scheme of B/X.

• on the other hand, since q is not a constant section of $\hat{E}/X$, neither is $p \in E(X)$, and the set $X^E_{tor} = \{\xi \in X(\mathbb{C}), p(\xi)$ is a torsion point on $E_\xi \simeq E_0\}$ is infinite. But by the lifting property, this set coincide the set $X^B_{tor} = \{\xi \in X(\mathbb{C}), \beta(\xi)$ is a torsion point on $B_\xi\}$. Therefore, the curve $Y = \beta(X) \subset B$ meets the set of torsion points of the various fibers of $B \to X$ Zariski-densely.

To perform the construction of β, we go to the generic fiber $B_x := B \otimes_S K := B_K$ of B/S, consider the non-constant point $q(x) := q_K \in \hat{E}_K = \hat{E}_0 \otimes_{\mathbb{C}} K$, and recall the construction of the Ribet point $\beta_K \in B_K(K)$ attached to q_K and to the antisymmetric isogeny $\varphi : \hat{E}_K \to E_K$, with transpose $\hat{\varphi} = -\varphi$. There are two ways to describe β_K :

(i) The first one [10] goes as follows : consider the pullback

$$\varphi^* B_K \in Ext(\hat{E}_K, \mathbb{G}_m) \text{ of } B_K \in Ext(E_K, \mathbb{G}_m)$$

under φ, and denote again by $\varphi : \varphi^* B_K \to B_K$ the natural extension of φ to $\varphi^* B_K$. Since B_K is parametrized by q_K, $\varphi^* B_K$ is parametrized by the point $\hat{\varphi}(q_K)$ of the dual E_K of $\hat{E}_K$. Now, choose an arbitrary K-rational point t_K in the fiber above q_K of the extension

[1] In particular, B/X is a semi-constant semi-abelian variety in the sense of [6]. However, the counter-example to Lindemann-Weierstrass given there is of a different nature; see also Remark 1.(ii) below.

$\varphi^* B_K$; in particular, its image $t^1_K := \varphi(t_K) \in B_K(K)$ satisfies $\pi_K(t^1_K) = \varphi(q_K)$, where $\pi_K = \pi(x) : B_K \to E_K$. The point t_K defines a one-motive $M_K : \mathbb{Z} \to \varphi^* B_K$, whose Cartier dual $\hat{M}_K : \mathbb{Z} \to B_K$ is given by a point $t^2_K \in B_K(K)$ projecting to $\hat{\varphi}(q_K) \in E(K)$. Finally, set $\beta_K = t^1_K - t^2_K$: this point of $B_K(K)$ is *independent of the choice of the auxiliary point* t_K above q_K (this is clear on the symmetric definition of duals given in Hodge III), and its image under π_K is the point $\varphi(q_K) - \hat{\varphi}(q_K) = 2\varphi(q_K) := p_K$ of $E_K(K)$.

(ii) The second one [4] is more geometric (and will actually not be used here): consider the Poincaré bundle $\mathcal{P}_K$ on $\hat{E}_K \times E_K$, rigidified above $(0, 0)$. Since φ is antisymmetric, the square of its restriction to the graph $\Phi_K \subset \hat{E}_K \times E_K$ of φ is trivial. Up to a 2- isogeny, we therefore get a unique non-zero K-regular section $\sigma_\varphi : \Phi_K \to \mathcal{P}_K|_{\Phi_K}$. Now B_K (plus a zero section) identifies with the restriction of $\mathcal{P}_K$ to $\{q_K\} \times E_K \simeq E_K$, and its fiber over $p'_K := \varphi(q_K)$ with the fiber of $\mathcal{P}_K$ over $(q_K, p'_K) \in \Phi_K(K)$. We then set $\beta'_K = \sigma_\varphi\big((q_K, p'_K)\big)$, and view β'_K as a point of $B_K(K)$ above p'_K. Up to multiplication by 2, this is the same point as the β_K above.

Restricting X if necessary, we can extend this point β_K to a section $\beta : X \to B$, which may be called the Ribet section attached to φ of the semi-abelian scheme B/X defined by the non-constant section q of $\hat{E}/X$ we had started with. More precisely, we can extend the auxiliary point t_K of the first construction to a section t of $\varphi^* B/X$, and repeat the whole process over X (minus some points), getting in particular a smooth one-motive M/X, sections t^1, t^2 over X, etc. By definition, $\pi \circ \beta$ is the section $p = 2\,\varphi \circ q$ of E/X extending p_K over X, and it remains to show that β satisfies the "lifting property".

So, let $\xi \in X(\mathbb{C})$ be a point such that $p(\xi)$ is a torsion point on the fiber E_ξ ($\simeq E_0$) of $E \to X$. We must show that $\beta(\xi)$ is a torsion point on the fiber B_ξ of $B \to X$. By the relation $p(\xi) = 2\varphi(q(\xi))$, $q(\xi)$ too is a torsion point on $\hat{E}_\xi$ (in passing, this shows that B_ξ is an isotrivial extension). So, among the points which lie on the fiber $(\varphi^* B)_\xi$ of $\varphi^* B/X$ above ξ (which is the pull-back

$$\varphi^* B_\xi \in Ext(\hat{E}_\xi, \mathbb{G}_m) \text{ of } B_\xi \in Ext(E_\xi, \mathbb{G}_m)\ ),$$

and which project to $q(\xi) \in (\hat{E}_\xi)_{tor}$, we now have not only the value $t(\xi)$ of the section $t : X \to \varphi^* B$ at ξ, but also plenty of torsion points of the complex semi-abelian variety $\varphi^* B_\xi$. Choose one of them, and call it $\tilde{t}_\xi$. Since $\tilde{t}_\xi$ and $t(\xi)$ differ by an element of $\mathbb{G}_m$, the first construction, whether applied to $t(\xi)$ or to $\tilde{t}_\xi$, will yield the *same* point $\beta_\xi \in B_\xi$, with $\pi(\beta_\xi) = p(\xi)$. Using $t(\xi)$, we see that $\beta_\xi = \beta(\xi)$; using the torsion point $\tilde{t}_\xi$ and the fact that $\hat{\varphi}(q(\xi))$ is a torsion point, we see that the weight filtrations of the corresponding complex one-motive $\tilde{M}_\xi$, hence of its dual $\hat{\tilde{M}}_\xi$, split up completely up to isogeny. Consequently, the points $\tilde{t}^1_\xi$, $\tilde{t}^2_\xi$ and β_ξ associated to $\tilde{t}_\xi$ by the first construction are all torsion points, and $\beta(\xi)$ is indeed a torsion point of B_ξ.

Remark 1 : i) (*from* X *to* $\hat{E}_0$) : let $\mathcal{B}$ be the "universal" extension of E_0 by $\mathbb{G}_m$, viewed as a group scheme over $\underline{Ext}^1(E_0, \mathbb{G}_m) \simeq \mathrm{Pic}^0_{E_0/\mathbb{C}} = \hat{E}_0$. The extension B attached to the section $q : X \to \hat{E}_0$ is the pull-back of $\mathcal{B}$ under q. Choosing $X = \hat{E}_0$, and $q =$ the identity

map, so that $K = \mathbb{C}(\hat{E}_0)$, we can therefore restrict to the case where $K = \mathbb{C}(\hat{E}_0)$ and q_K is the generic point of $\hat{E}_0(K)$. The Appendix - and most of §2 - concerns this generic case

$$X = \hat{E}_0, q = id, B = \mathcal{B}.$$

ii) *(when $g > 1$)* : Ribet sections β can be defined over any abelian scheme A/X, of relative dimension g, which admits an antisymmetric isogeny $\varphi : \hat{A} \to A$. If $g = 1$, this forces A be to be iso-constant, hence the $E_0/\mathbb{C}$ above. But as soon as $g > 1$, there are examples of simple non constant A/X with such a $\varphi = -\hat{\varphi}$. The section β attached to φ and to a section $q \in \hat{A}(X)$ will again satisfy the "lifting property". However, in order to ensure that the set $X^A_{tor} = \{\xi \in X(\mathbb{C}), \pi \circ \beta(\xi)$ is a torsion point on $A_\xi\}$ be infinite, one must in general insist that $dim(X) \geq g$. So, the counterexample does not extend to extensions by $\mathbb{G}_m$ of higher dimensional abelian schemes *over curves.*

2 ... in support of Pink's general conjecture.

In [13], R. Pink mentions the similarity of Conjecture 6.2 with Y. André's result on special points on elliptic pencils [2], III, p. 9. Viewing the scheme B/X above as a "semi-abelian pencil" over the fixed elliptic curve E_0, one could define its special points as the torsion points lying on a fiber which is an isotrivial extension. As mentioned in passing during the proof of the lifting property, the curve $Y = \beta(X)$ even contains infinitely many special points in this sense.

Going further in this direction, we will now construct a mixed Shimura variety $S(\varphi)$ into which the image $Y = \beta(X)$ of the Ribet section β can be mapped in a natural way. Denoting this map by $i : Y \to S(\varphi)$, we have under the hypotheses of §1 (or more generally, of Footnote (2) below) :

Theorem 2. *The algebraic subvariety $Z = i(Y)$ of the mixed Shimura variety $S(\varphi)$ passes through a Zariski-dense set of special points of $S(\varphi)$, and is indeed a special subvariety of $S(\varphi)$.*

This, of course, is in full concordance with the prediction of the general Conjecture 1.3 of [13] (more specifically, of the case $d = 0$ of Conjecture 1.1).

Here, I will merely give a set-theoretic description of the construction of $S(\varphi)$. We fix an integer $g \geq 1$ and a totally imaginary quadratic integer $\alpha = -\overline{\alpha}$, and denote by S_0 (a component of) the pure Shimura variety parametrizing abelian varieties A endowed with a principal polarization $\psi : \hat{A} \to A$ (in particular, $\psi = \hat{\psi}$), with some level structure, and with an embedding $j : \mathbb{Z}[\alpha] \to \mathrm{End}(A)$ such that $\psi \circ \widehat{j(\alpha)} \circ \psi^{-1} = j(\overline{\alpha})$. Let

$$\big(\mathcal{A},\ \psi,\ \varphi := j(\alpha) \circ \psi\big)$$

be the corresponding universal abelian scheme over S_0 (in particular, $\varphi = -\hat{\varphi} : \hat{\mathcal{A}} \to \mathcal{A}$ is antisymmetric). Then, $S_1 := \mathcal{A} \times_{S_0} \hat{\mathcal{A}}$ is a mixed Shimura variety parametrizing one-motives of the shape $M : \mathbb{Z} \to A \times \hat{A}$, i.e. couples of points $(p, q) \in A \times \hat{A}$, with $\{A\} \in S_0$,

and we can view
$$S_1(\varphi) = \{(A, p, q) \in S_1, p = 2\varphi(q)\}$$
as a mixed Shimura subvariety of S_1. Finally, consider the Poincaré bi-extension
$$\varpi = (\varpi_1, \varpi_2) : \mathcal{P}^* \to \mathcal{A} \times_{S_0} \hat{\mathcal{A}}.$$
This is a $\mathbb{G}_m$-torsor over $S_1 = \mathcal{A} \times_{S_0} \hat{\mathcal{A}}$, which can again be viewed as a mixed Shimura variety, now parametrizing one-motives $M : \mathbb{Z} \to B$ of constant and toric ranks equal to 1, where on denoting by $\big(A, p := \varpi_1(M), q := \varpi_2(M)\big)$ the point defined by $\varpi(M)$ in S_1, $B = B_q$ is the extension of A by $\mathbb{G}_m$ attached to q, while the image b of $1 \in \mathbb{Z}$ is a point on B projecting to $p \in A$. Breaking the symmetry between $\mathcal{A}$ and $\hat{\mathcal{A}}$, we can alternatively consider $\hat{\mathcal{A}} \simeq \underline{Ext}^1_{S_0}(\mathcal{A}, \mathbb{G}_m)$ as a mixed Shimura variety, and view $\varpi_2 : \mathcal{P}^* \to \hat{\mathcal{A}}$ as the "universal" extension $\mathcal{B}$ of $\mathcal{A}$ by $\mathbb{G}_m$, over its parameter space $\hat{\mathcal{A}}$. We at last define
$$S(\varphi) := \varpi^{-1}(S_1(\varphi))$$
as the mixed Shimura subvariety of $\mathcal{P}^* \simeq \mathcal{B}$ whose points parametrize one-motives $M : \mathbb{Z} \to B$ such that $\varpi_1(M) = 2\varphi(\varpi_2(M))$.

We now consider an abelian scheme A/X of the type parametrized by S_0, over some irreducible algebraic variety $X/\mathbb{C}$. There then exists a unique morphism $i_0 : X \to S_0$ such that A/X is the pull-back of $\mathcal{A}$ under i_0. We fix a section $q : X \to \hat{A}$, corresponding to a semi-abelian scheme $\pi : B \to A$ over X, and perform Ribet's construction[2], yielding sections $p = 2\varphi \circ q : X \to A,\ \beta : X \to B$, with $p = \pi \circ \beta$. By the universal property of $S(\varphi)$, there exists a unique morphism
$$i : X \to S(\varphi)$$
above i_0 such that the smooth X-one-motive $M : \mathbb{Z} \to B$ defined by β is the pull-back under i of the universal one-motive $\mathcal{M} : \mathbb{Z} \to \mathcal{B}$. We again denote by $i : B \to S(\varphi) \subset \mathcal{B}$ the extension of i to B/X. The image $Z := i(Y) \subset S(\varphi)$ of $Y = \beta(X) \subset B$ is the algebraic subvariety of $S(\varphi)$ to be studied for Theorem 2.

We first check that Z contains a Zariski-dense set of special points of $S(\varphi)$. By definition, these points represent complex one-motives such that the underlying abelian variety is CM, and whose weight filtrations are totally split up to isogeny. By the first hypothesis

[2] When $g = 1$ as in the first Section, S_0 is reduced to a CM point $\{E_0\}$. Taking into account Remark 1.(ii), we are now proving Theorem 2 for any $g \geq 1$. However, in the case $g > 1$, we must add the following hypotheses to its statement : the base $X/\mathbb{C}$ is an irreducible variety of dimension $\geq g$, and

(a) the image $i_0(X)$ meets the set of CM points of S_0 Zariski-densely.

(b) $q(X)$ meets the set of torsion points of the various CM fibers of $\hat{A}$ Zariski-densely;

For the sake of simplicity, we will *assume*, in what follows, *that* $i_0(X)$ *is a Shimura subvariety of* S_0 (more or less $\Leftrightarrow$ (a) under André-Oort), and *that* $q : X \to \hat{A}$ *dominates the "generic case"* $id : \hat{A} \to \hat{A}$ ($\Rightarrow$ (b)). Furthermore, the latter hypothesis implies the useful (but not necessary) property that :

(c) q factors through no proper closed subgroup scheme of $\hat{A}/X$.

made in Footnote [(2)], the projection $i_0(X)$ of Z to S_0 passes through a Zariski-dense set of CM points. In the fiber of any such point, the projection of Z to $S_1(\varphi)$ passes through a Zariski-dense set of torsion points $(p = 2\varphi(q), q)$, because of the second hypothesis made in this foot-note. The lifting property established in §1 (more accurately, the sharper version mentioned in passing) now shows that Z does meet the set of special points of $S(\varphi)$ Zariski-densely.

It remains to show that Z is a special subvariety of $S(\varphi)$. We will check this Hodge-theoretically. Denote by G, resp. P, the generic Mumford-Tate group of the Shimura variety $i_0(X)$, resp. of the mixed Shimura subvariety of $S(\varphi)$ lying above $i_0(X)$. Then, P is the semi-direct product of its unipotent radical $W_{-1}(P)$ by G, and $W_{-1}(P)$ is the semi-direct product of $W_{-2}(P) \subset \mathbb{G}_a$ by the vectorial group $Gr_{-1}(P) \subset \mathbb{G}_a^{2g}$: notice that $Gr_{-1}(P)$ is the unipotent radical of the generic Mumford-Tate group of $S_1(\varphi)$, and the inclusion $Gr_{-1}(P) \subset \mathbb{G}_a^{2g}$ follows from the linear dependence relation $p = 2\varphi(q)$. Similarly, let $P_z \subset P$ denote the Mumford-Tate group of a sufficiently general point z in Z; then, P_z is the semi-direct product of its unipotent radical $W_{-1}(P_z)$ by G, and $W_{-1}(P_z)$ is the semi-direct product of $W_{-2}(P_z) \subset W_{-2}(P)$ by the vectorial group $Gr_{-1}(P_z) \subset Gr_{-1}(P)$.

Now, by the second hypothesis of the footnote, $\mathbb{Z}.q(X)$ is Zariski-dense in $\hat{A}$, and Proposition 1 of [1] shows that $Gr_{-1}(P_z) = Gr_{-1}(P) = \mathbb{G}_a^{2g}$. On the other hand, Theorem 1 of [4] shows that $W_{-2}(P) = \mathbb{G}_a$, while $W_{-2}(P_z) = \{0\}$, and more precisely, that *for any point $s \in S(\varphi)(\mathbb{C})$ whose Mumford-Tate group P_s satisfies $Gr_{-1}(P_s) = \mathbb{G}_a^{2g}$, we have*

$$W_{-2}(P_s) = \{0\} \Leftrightarrow \exists\, \tilde{s} \in [s],\ \tilde{s} \in Z,$$

where $[s]$ denotes the Hecke orbit of s. In other words, up to isogenies, the points of Z are characterized by the existence of an exceptional Hodge tensor in their Betti realization, which does not exist at the generic point of $S(\varphi)$. So, Z is indeed a special subvariety of $S(\varphi)$.

Remark 2 : (i) This section shows that the special subvarieties of the mixed Shimura variety $\mathcal{B}$ do not necessarily correspond to families of semi-abelian subvarieties, so that Theorem 5.7 and 6.3 of [13] must be modified. Similarly, the mixed Shimura varieties of Hodge type $\{W_{-2}(P) = 0\}$ are not necessarily Kuga fiber varieties (compare [12], Example 1.10, and [4], Remark (v)).

(ii) The strange divisibility properties of Ribet points alluded to in the introduction of §1 are precisely reflected by the ℓ-adic analogue of the vanishing of $W_{-2}(P_z), z \in S(\varphi)(\overline{\mathbb{Q}})$; cf. [4], Theorem 1.(ii). Actually, K. Ribet gives in [14] an explicit description of the exceptional Galois invariant tensor occuring in the ℓ-adic cohomology of these one-motives, which applies to all their realizations.

iii) We refer to [3] for a complete description of the unipotent radical $W_{-1}(P)$ of the Mumford-Tate group of one-motives of higher toric or constant ranks. When both ranks are equal to 1, the fact that $W_{-2}(P)$ vanishes in the case of antisymmetrically self-dual one-motives is an exercise in goup theory, cf. [5], Lemme 6.

3 Appendix, by Bas Edixhoven

We go back to the setting and the notations $E_0, X, q, \varphi = -\hat{\varphi}, \pi : B \to E$ of § 1, but to make later comparisons easier, we henceforth consider the extension

$$B = B_{2q}$$

of E by $\mathbb{G}_m$ given by $2q \in \hat{E}(X)$, and denote by β_R in $B_{2q}(X)$ the Ribet section which the construction of § 1 attaches to the data $\{2q, \varphi\}$; in particular, the projection $p := \pi \circ \beta_R = \varphi(2q) - \hat{\varphi}(2q) = 2\varphi(2q))$ of β_R on $E(X)$ is now twice the section considered in §1.

Interpreting B_{2q}/X as the generalized jacobian $\mathrm{Pic}^0_{C/X}$ of a singular curve C/X with normalization $\hat{E} = \mathrm{Pic}^0_{E/X} = \hat{E}_0 \times X/X$, we will here construct[3] a concrete section β_J (with "J" for "Jacobian") of B_{2q}/X, which enjoys all the properties[4] of the Ribet section β_R, and thereby provides a new proof of Theorem 1, but for which the "lifting property" takes the following sharper form :

Theorem 3. *For any $\xi \in X$, the image of $\beta_J(\xi)$ under $\pi : B_\xi \to E_\xi \simeq E_0$ satisfies $\pi(\beta_J(\xi)) = \pi(\beta_R(\xi)) := p(\xi)$. And if $p(\xi)$ is a torsion point of E_0 of order n, with n prime to $2\deg(\varphi)\deg(\varphi+\psi)\deg(\varphi-\psi)$, then $\beta_J(\xi)$ is a torsion point of B_ξ, of order dividing n^2.*

Here, $\psi = \hat{\psi} : \hat{E} \to E$ denotes the standard principal polarization: its inverse sends a point P to the class of the divisor $(P) - (0)$. Without loss of generality, we assume that we are in the generic case $X = \hat{E}_0, q = \mathrm{id}$, with $K := \mathbb{C}(X) = \mathbb{C}(\hat{E}_0)$, but we keep to the notation X to indicate that some points of X may have to be removed in the constructions which follow. We denote by $Q = \psi \circ q : X \to E$ the section of E/X such that $q \in \hat{E}(X) = \mathrm{Pic}^0(E/X)/\mathrm{Pic}(X)$ is represented by the divisor $(Q) - (0)$ on E. By biduality, the section Q in $E(X) = \mathrm{Pic}^0(\hat{E}/X)/\mathrm{Pic}(X)$ is represented by the divisor $(q) - (0)$ on $\hat{E}$.

Let C/X be the singular curve over X obtained by identifying the disjoint sections q and $-q$ of $\hat{E}$ (we remove $\hat{E}_0[2]$ from X). As a set, it is the quotient of $\hat{E}$ by the equivalence relation generated by $q(\xi) \sim -q(\xi)$ with ξ ranging over X. The topology on C is the finest one for which the quotient map quot: $\hat{E} \to C$ is continuous: a subset U of C is open if and only if $\mathrm{quot}^{-1}U$ is open in $\hat{E}$. The regular functions on an open set U of C are the regular functions f on $\mathrm{quot}^{-1}U$ such that $f(q(\xi)) = f(-q(\xi))$ whenever $\mathrm{quot}(q(\xi))$ is in U. It is proved in Thm. 5.4 of [9] that this topological space with sheaf of $\mathbb{C}$-valued functions is indeed an algebraic variety over $\mathbb{C}$. In categorical terms, quot: $\hat{E} \to C$ is the equalizer of the pair of morphisms $(q, -q)$ from X to $\hat{E}$.

The curve $C \to X$ is a family of singular curves, each with an ordinary double point; it is semi-stable of genus two (see [8, 9.2/6]). Its normalization is quot: $\hat{E} \to C$. Its

[3] I thank Lenny Taelman for the suggestion to pay more attention to the symmetry between the points q and $-q$ to be identified and the divisor β_a that gives the section β_J.

[4] In particular, β_J does not factor through any proper closed subgroup scheme of B. For a discussion relating β_J and β_R, see Remark 3.ii below.

generalized jacobian $B := \mathrm{Pic}^0_{C/X}$ is described in [8], 8.1/4, 8.2/7, 9.2/1, 9.3/1. As $C \to X$ has a section (for example $\bar{0} := \mathrm{quot} \circ 0$ and $\bar{q} := \mathrm{quot} \circ q$), we have, for every $T \to X$, that $B(T) = \mathrm{Pic}^0(T \times_X C/T)/\mathrm{Pic}(T)$, where $\mathrm{Pic}^0(T \times_X C/T)$ is the group of isomorphism classes of line bundles on $T \times_X C$ that have degree zero on the fibres of $T \times_X C \to T$. The group $\mathrm{Pic}(T)$ is contained as direct summand in $\mathrm{Pic}^0(T \times_X C/T)$ via pullback by the projection $T \times_X C \to X$ and a chosen section. In particular, a divisor D on C that is finite over X, disjoint from $\bar{q}(X)$ and of degree zero after restriction to the fibers of $C \to X$ gives the invertible $\mathcal{O}_C$-module $\mathcal{O}_C(D)$ that has degree zero on the fibers and therefore gives an element denoted $[D]$ in $B(X)$.

For ξ in X, the fiber B_ξ is, as abelian group, the group $\mathrm{Pic}^0(C_\xi)$. In terms of divisors this is the quotient of the group $\mathrm{Div}^0(C_\xi)$ of degree zero divisors with support outside $\{\bar{q}(\xi)\}$ by the subgroup of principal divisors $\mathrm{div}(f)$ for nonzero rational functions f in $\mathbb{C}(C_\xi)^\times$ that are regular at $\bar{q}(\xi)$. As $C_\xi - \{\bar{q}(\xi)\}$ is the same as $\hat{E}_0 - \{q(\xi), -q(\xi)\}$, $\mathrm{Div}^0(C_\xi)$ is the group of degree zero divisors on $\hat{E}_0$ with support outside $\{q(\xi), -q(\xi)\}$. An element f of $\mathbb{C}(C_\xi)^\times$ that is regular at $\bar{q}(\xi)$ is an element of $\mathbb{C}(\hat{E}_0)^\times$ that is regular at $q(\xi)$ and $-q(\xi)$ and satisfies $f(q(\xi)) = f(-q(\xi))$. This gives us a useful description of B_ξ.

The normalization map $\mathrm{quot}\colon \hat{E} \to C$ induces a morphism of group schemes over X

$$\pi\colon B = \mathrm{Pic}^0_{C/X} \to \mathrm{Pic}^0_{\hat{E}/X} = E/X,$$

and identifies B with the extension of E by $\mathbb{G}_m$ given by the section $2q \in \hat{E}(X)$. For ξ in X, the class $[\delta]$ in B_ξ of a divisor $\delta \in \mathrm{Div}^0(C_\xi)$ lies in the kernel $\mathbb{C}^\times$ of π_ξ if and only if there exists $f \in \mathbb{C}(\hat{E}_0)^\times$ such that $\delta = \mathrm{div}(f)$ on $\hat{E}_0$, and it is then a torsion point in $\mathbb{C}^\times$ if and only if the quotient $f(q(\xi))/f(-q(\xi)) \in \mathbb{C}^\times$, which does not depend on the choice of f, is a root of unity.

We recall that for an elliptic curve $\mathcal{E}$ over an algebraically closed field k, $\psi\colon \hat{\mathcal{E}} \to \mathcal{E}$ the standard polarization and u in $\mathrm{End}(\mathcal{E})$, the pullback map u^* on $\mathrm{Div}(\mathcal{E})$ induces $\hat{u}$ in $\mathrm{End}(\hat{\mathcal{E}})$, the dual of u, and then $\bar{u} := \psi\hat{u}\psi^{-1}$ in $\mathrm{End}(\mathcal{E})$ is called the Rosati-dual of u; it is characterized by the property that in $\mathrm{End}(\mathcal{E})$ we have $\bar{u}u = \deg(u)$ and $u + \bar{u} \in \mathbb{Z}$. Also, the pushforward map u_* on $\mathrm{Div}(\mathcal{E})$ induces an element still denoted u_* in $\mathrm{End}(\hat{\mathcal{E}})$ such that $\psi^{-1}u = u_*\psi^{-1}$ and u_*u^* is multiplication by $\deg(u)$ in $\mathrm{End}(\hat{\mathcal{E}})$. Hence u_* is the Rosati dual of u^*. We have $\hat{\hat{u}} = u$, $\bar{\bar{u}} = u$ and $\bar{u}_* = u^*$ in $\mathrm{End}(\hat{\mathcal{E}})$. For f a nonzero rational function on $\mathcal{E}$ and $u \neq 0$ we have $u^*\mathrm{div}(f) = \mathrm{div}(f \circ u)$, and $u_*\mathrm{div}(f) = \mathrm{div}(\mathrm{Norm}_u(f))$, where $\mathrm{Norm}_u\colon k(\mathcal{E})^\times \to k(\mathcal{E})^\times$ is the norm map along u. Of course, all this applies to E_0 and $\hat{E}_0$ over $\mathbb{C}$, and to E and $\hat{E}$ over the algebraic closure of the function field K of X.

We will use Weil reciprocity: for f and g nonzero rational functions on a nonsingular irreducible projective curve $\mathcal{E}$ over an algebraically closed field k such that $\mathrm{div}(f)$ and $\mathrm{div}(g)$ have disjoint supports, one has $f(\mathrm{div}(g)) = g(\mathrm{div}(f))$, where for $D = \sum_P D(P){\cdot}P$ a divisor on $\mathcal{E}$ one defines $f(D) = \prod_P f(P)^{D(P)}$, cf. [15], III, Prop. 7. In Remark 3.(i) after the proof, we will also use the Weil pairing on $\mathcal{E}$. For n a positive integer and P and Q in $\mathrm{Pic}^0(\mathcal{E})[n]$ the element $e_n(P,Q)$ in $\mu_n(k)$ is defined as follows. Let D_P and D_Q in $\mathrm{Div}^0(\mathcal{E})$ be disjoint divisors representing P and Q. Let f and g be in $k(\mathcal{E})^\times$ such that

$nD_P = \mathrm{div}(f)$ and $nD_Q = \mathrm{div}(g)$. Then $e_n(P,Q) = f(D_Q)/g(D_P)$. For n invertible in k this pairing e_n is a perfect alternating pairing.

To define $\beta_J = \beta_J(a)$, we let $a \in \mathrm{End}(\hat{E}_0)$ be *any* endomorphism such that $a^3 - a \neq 0$. We set $\alpha := \hat{a}$ in $\mathrm{End}(E_0)$ and $\varphi := \alpha \circ \psi \colon \hat{E}_0 \to E_0$. Just as in [10], we will not need that $\hat{\varphi} = -\varphi$. However, if $\hat{\varphi} = \varphi$ then $\pi\beta_J$ will be zero in $\hat{E}(X)$, so we will insist that $\hat{\varphi} - \varphi \neq 0$. For s in $\hat{E}(X)$ we let (s) denote the relative divisor that it gives on $\hat{E}$, and, if (s) is disjoint from the singular locus $\bar{q}(X)$ of C, also the relative divisor on C that it gives.

We set:

$$\beta_a := a^*\big((q) - (-q)\big) - a_*\big((q) - (-q)\big) \quad \text{in } \mathrm{Div}^0(\hat{E}/X). \tag{1}$$

The reader should note the antisymmetric use of both a^* and a_* in the definition of β_a. To get its support disjoint from $\bar{q}(X)$ we remove from X the finite set $\ker(a^2 - 1)$: note that, for $\xi \in X$, $\beta_a(\xi)$ and $(q(\xi)) - (-q(\xi))$ are not disjoint if and only if $aq(\xi) = q(\xi)$ or $aq(\xi) = -q(\xi)$. We can now also view β_a as element of $\mathrm{Div}^0(C/X)$, and we set:

$$\beta_J := [\beta_a] \quad \text{in } B(X).$$

The image of $\pi\beta_J$ of β_J in $E(X)$ is the class of the divisor β_a on $\hat{E}$, hence we have, on denoting by $\simeq$ linear equivalence on $\mathrm{Div}^0(\hat{E}/X)$:

$$\begin{aligned}
\pi\beta_J &= \bar{a}_*\big((q) - (-q)\big) - a_*\big((q) - (-q)\big)\\
&\simeq (\bar{a}q) - (\bar{a}(-q)) - ((aq) - (a(-q))\\
&\simeq (2(\bar{a} - a)q) - (0) = 2(\alpha - \bar{\alpha})Q \quad \text{in } E(X).
\end{aligned}$$

In particular, since $\bar{a} - a$ is nonzero, then $\pi\beta_J$ is not torsion in $E(X)$, and in fact

$$\pi\beta_J = \big((2\bar{a}q) - (0)\big) - \big((2aq) - (0)\big) = \varphi(2q) - \hat{\varphi}(2q) = p = \pi\beta_R,$$

as was to be checked for the first part of Theorem 3.

Now we start the proof of the second part of Theorem 3. Let n be a positive integer prime to $2\deg(\varphi)\deg(\varphi + \psi)\deg(\varphi - \psi) = 2\deg(a(a^2 - 1))$, and let $\xi \in X$ be a point such that $p(\xi) \in E_\xi = E_0$ is a torsion point of order n. Actually, we will *assume that $q(\xi)$ is a point of order n in $\hat{E}_\xi = \hat{E}_0$* : in the antisymmetric case considered in Theorem 3, we have $p(\xi) = 4\varphi(q(\xi))$, and the two conditions are equivalent by the primality assumption. To ease notations, we now drop the mention of ξ, writing $E, B, q, \ldots$ instead of $E_\xi, B_\xi, q(\xi), \ldots$.

As $nq = 0$ in $\hat{E}_0$, we have $n\pi\beta_J = 0$ in E_0. This means that $n\beta_a$ is a principal divisor on $\hat{E}_0$. Let $f \in \mathbb{C}(\hat{E}_0)^\times$ be such that $\mathrm{div}(f) = n(q) - n(-q)$ in $\mathrm{Div}(\hat{E}_0)$. Then we have, on $\hat{E}_0$:

$$\begin{aligned}
\mathrm{div}(f \circ a) &= a^*\mathrm{div}(f) = a^*\left(n(q) - n(-q)\right),\\
\mathrm{div}(\mathrm{Norm}_a(f)) &= a_*\mathrm{div}(f) = a_*\left(n(q) - n(-q)\right).
\end{aligned}$$

We define:

$$g_a := (f \circ a)/\mathrm{Norm}_a(f) \quad \text{in } \mathbb{C}(\hat{E}_0)^\times.$$

Then we have:

$$n\beta_a = \mathrm{div}(f \circ a) - \mathrm{div}(\mathrm{Norm}_a(f)) = \mathrm{div}(g_a) \quad \text{on } \hat{E}_0.$$

This means that $n[\beta_J]$ in B is the element $g_a(q)/g_a(-q)$ of $\mathbb{C}^\times$. As the divisor of f has support disjoint from that of g_a and of $a^*\mathrm{div}(f)$ and $a_*\mathrm{div}(f)$, Weil reciprocity gives us:

$$\begin{aligned}\left(\frac{g_a(q)}{g_a(-q)}\right)^n &= g_a(\mathrm{div}(f)) = f(\mathrm{div}(g_a)) = f(\mathrm{div}(f \circ a) - \mathrm{div}(\mathrm{Norm}_a(f)))\\ &= \frac{f(\mathrm{div}(f \circ a))}{f(\mathrm{div}(\mathrm{Norm}_a(f)))} = \frac{(f \circ a)(\mathrm{div}(f))}{f(a_*\mathrm{div}(f))} = \frac{f(a_*\mathrm{div}(f))}{f(a_*\mathrm{div}(f))} = 1.\end{aligned}$$

This finishes the proof of Theorem 3.

Remark 3 : i) We have shown that for ξ torsion in $X \subset \hat{E}_0$ and n its order, we have $n^2\beta_J(\xi) = 0$ in B_ξ. We will show that for $n > 1$ odd and prime to $\deg((a^2-1)(\bar{a}-a))$ there exist ξ in X with $p(\xi)$ of order n in E_0 such that the order of $\beta_J(\xi)$ equals n^2.

Let n be such an integer. Recall the notation $Q = \psi(q)$, and let ξ in $\hat{E}_0$ be of order n such that $e_n(2(\alpha - \bar{\alpha})Q(\xi), 2Q(\xi))$ is of order n in $\mathbb{C}^\times$. Such ξ exist because $\alpha - \bar{\alpha}$ is an automorphism of $E_0[n]$ that is not scalar multiplication by an element of $\mathbb{Z}/n\mathbb{Z}$. And such a ξ is in X because n is prime to $2\deg(a^2-1)$. By construction, $p(\xi) = 2(\alpha - \bar{\alpha})Q(\xi)$ is represented by the divisor $\beta_a(\xi)$, and $2Q(\xi)$ is represented by $(q(\xi)) - (-q(\xi))$. Again, let us drop the ξ's from our notation. Then $n\beta_a = \mathrm{div}(g_a)$, and $n((q) - (-q)) = \mathrm{div}(f)$. We first compute:

$$\begin{aligned}f(\beta_a) &= f(a^*((q) - (-q)) - a_*((q) - (-q))) = \frac{(\mathrm{Norm}_a f)((q) - (-q))}{(f \circ a)((q) - (-q))}\\ &= g_a^{-1}((q) - (-q)) = \frac{g_a(-q)}{g_a(q)}.\end{aligned}$$

Hence:

$$e_n(2(\alpha - \bar{\alpha})Q, 2Q) = \frac{g_a((q) - (-q))}{f(\beta_a)} = \frac{g_a(q)}{g_a(-q)f(\beta_a)} = \left(\frac{g_a(q)}{g_a(-q)}\right)^2.$$

ii) The Ribet section β_R and the present section β_J of B/X differ by an element of $\mathbb{G}_m(X)$, and it is natural to ask whether they are actually equal. In this respect, notice that by Theorem 3, $\beta_J(X)$ passes through a Zariski-dense set of special points of the mixed Shimura variety $S(\varphi)$ of §2. If we assume Pink's general Conjecture 1.3 of [13], $\beta_J(X)$ must then be a component of the Hecke orbit of the special subvariety Z defined by $\beta_R(X)$. Hence, at least conjecturally, the section $\beta_R - \beta_J$ of B/X is a root of unity.

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